Engineering Probability and Statistics

Course Description

- This is an introductory calculus-based course to probability and statistics intended for engineering students. During the course, real-world problems, drawn from different engineering disciplines, will be modeled and solved using basic probability principles. An introduction to engineering statistics and estimation theory will also be provided.

Main goals for the course

Upon the successful completion of this course, a student should:

- Understand the fundamental concepts of probability theory.
- Understand the basic probability principles concerning single and multiple random variables as well as the operations on these random variables.
- Learn the common discrete and continuous probability distribution functions and their statistical properties.
- Be able to use the probability distributions in solving real-world engineering problems.
- Understand the basic principles of statistics and statistical estimation.
- Understand the main probabilistic concepts in estimation theory and engineering decision making.

Detailed Course Outline

- **Fundamental Concepts of Probability**: set operations, definitions of probability, discrete and continuous probability functions, conditional probability and independence, theorem of total probability, Bayes' Theorem, counting techniques.
- **Single Random Variables and Probability Distributions**: the random variable concept, probability mass and probability density functions, cumulative distribution function, common continuous and discrete density functions (uniform, exponential, Gaussian, binomial, geometric, hyper-geometric, Poisson process), expectation, transformations of discrete and continuous random variables.
- **Probability Distributions for Multiple Random Variable:** joint probability density/mass function, marginal density functions, statistical independence, the convolution integral
- jointly Gaussian random variables, transformation of multiple random variables, linear transformation of Gaussian random variables.
- **Elementary Statistics**: Sample mean and sample variance, regression techniques (linear, polynomial, exponential), the central limit theorem.
- **Estimation Theory and Applications**: desirable qualities of point estimators, maximum likelihood estimator, interval estimators for the mean (variance known, variance unknown) and variance (mean known, mean unknown), interval estimation of the binomial proportion.
- **Engineering Decisions**: Bayes' hypothesis testing (decision strategy) and illustrative examples, classical decision theory, type I and type II errors, one-sided and two-sided hypothesis testing, the operating characteristic curve.

Why Engineers Study Probability and Statistics?

- Probability models are used in situations where a deterministic model is not applicable.
- Engineers study probability because uncertainties are unavoidable in the design and planning of engineering systems.
- In a mechanical system, components fail probabilistically because material defects cannot be controlled 100%. Knowledge of the probabilistic model of components enables the engineer to design a system with high reliability (reliability describes the ability of a system or component to function under stated conditions for a specified period of time, Wikipedia).
- In electrical engineering, the performance of a communication system highly depends on the noise that impairs the information signal. Statistical methods that model the noise are used to design optimum systems.
- If you know how likely a component or system is to fail after collecting enough data, you can make recommendations regarding the warrantee period given to sold components and appliances.
- In aerospace engineering, the turbulence that hits a plane is random phenomenon that depends on changes in pressure and temperature. These two factors change relatively suddenly and cannot be anticipated completely. Probabilistic models are needed for a proper design.
- The probability of failure of a structure is a question that can be answered only using probabilistic models.
- Quality control and process control use statistics as a tool to manage conformance to specifications of manufacturing processes and their products (Wikipedia)
- System identification uses statistical methods to build mathematical models of dynamical systems from measured data. System identification also includes the optimal design of experiments for efficiently generating informative data for fitting such models (Wikipedia)
- Chemical engineers use probability and statistics to assess experimental data and control and improve chemical processes.

Basic Definitions

We start our treatment of probability theory by introducing some basic definitions.

Experiment: By an experiment, we mean any procedure that:

1. Can be repeated, theoretically, an infinite number of times.
2. Has a well-defined set of possible outcomes.

Sample Outcome: Each of the potential eventualities of an experiment is referred to as a sample outcome(s).

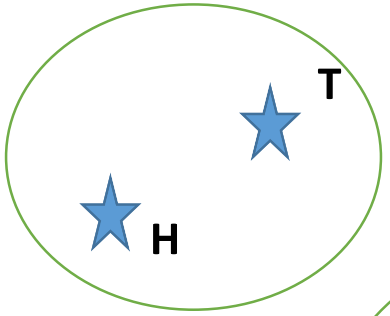
Sample Space: The totality of sample outcomes is called the sample space (S).

Event: Any designated collection of sample outcomes, including individual outcomes, the entire sample space and the null space, constitute an event.

Occurrence: An event is said to occur if the outcome of the experiment is one of the members of that event.

Examples

- **Example** : Consider the experiment of flipping a coin once.
- **Solution**: The sample space consists of two elements : S = {H, T}.



- **Example:** Consider the experiment of rolling a die.

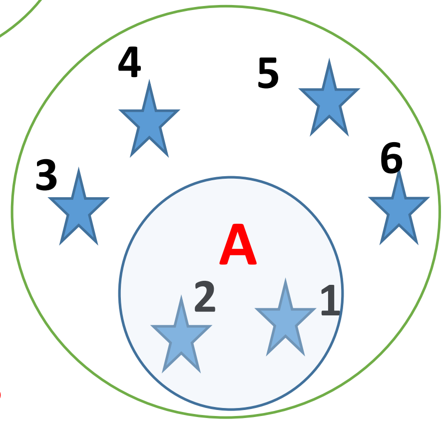
a. Find the sample space

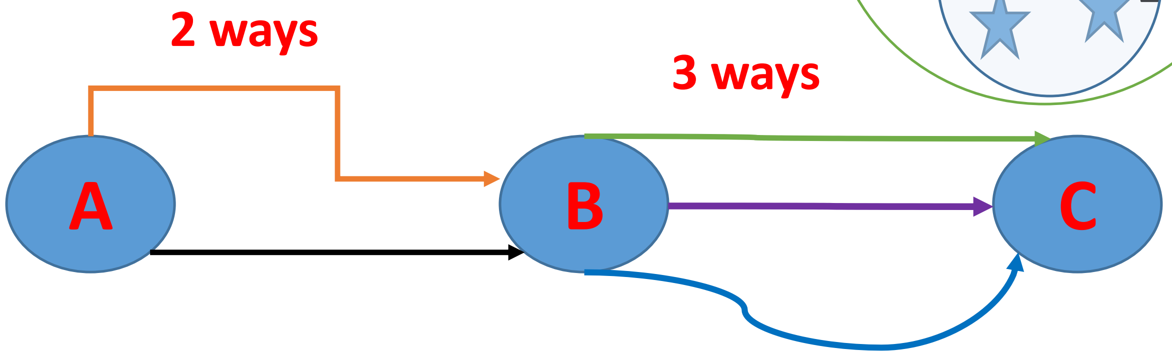
b. Find the event A= {outcome value < 3}

- **Solution:**

a. S = {1, 2, 3, 4, 5, 6}

b. Event A= {1, 2}





- **Example** : Towns A and B are connected by two roads and towns B and C are connected by 3 roads. By how many routes can one travel from town A to town C?
- **Solution**: The sample space S consists of 6 possible routes.

Examples

Example : Consider the experiment of flipping a coin three times.

1	2	3
(H,T)	(H,T)	(H,T)

a. What is the sample space?

b. Which sample outcomes make up the event: A = {Majority of coins show heads}.

Solution:

a- Sample Space: S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}; 8 possible outcomes

b- A = {HHH, HHT, HTH, THH}; 4 elements

Example: A die and a coin are flipped simultaneously.

(H,T)

- Find the sample space
- Find the event A= {number divisible by 2, Head}

1	2	3	4	5	6

Solution:

- a. Sample Space S={(1, H), (2, H), (3, H), (4, H), (5, H), (6, H), (1, T), (2, T), (3, T), (4, T), (5, T), (6, T)}; 12 pair of elements.
- A = {(2, H), (4, H), (6, H)}; 3 pair of elements

Algebra of Events

Let A and B be two events defined over the sample space S, then:

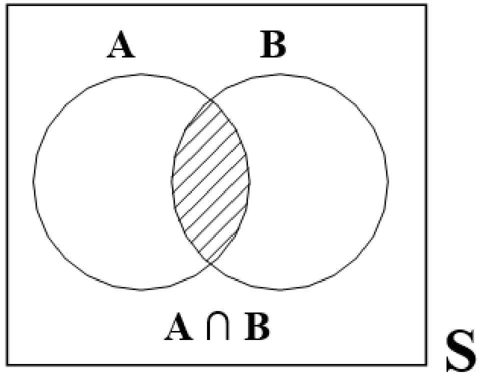
- The *intersection* of A and B, ($A \cap B$), is the event whose outcome belongs to both A and B.
- The *union* of A and B, ($A \cup B$), is the event whose outcome belongs to either A or B or both.
- Events A and B are said to be *Mutually Exclusive* (or disjoint) if they have no outcomes in common, that is $A \cap B = \emptyset$, where $\emptyset$ is the null set (a set which contains no outcomes).
- The *complement* of A (A^c or $\bar{A}$) is the event consisting of all outcomes in S other than those contained in A.

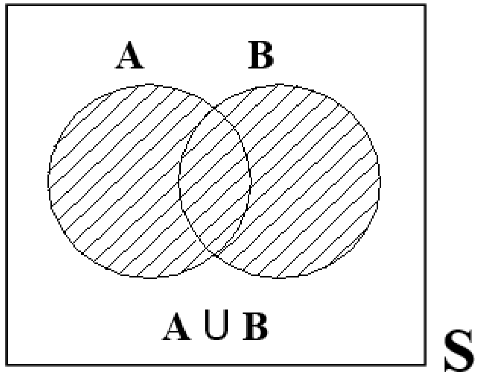
Venn Diagram: is a graphical format often used to simplify the manipulation of complex events.

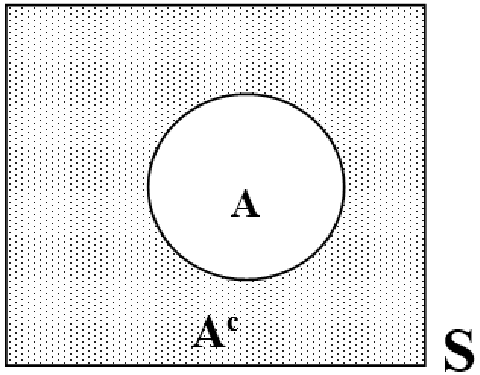
De Morgan's Laws:

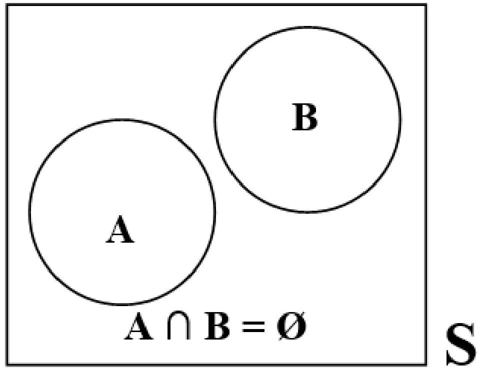
Use Venn diagrams to show that:

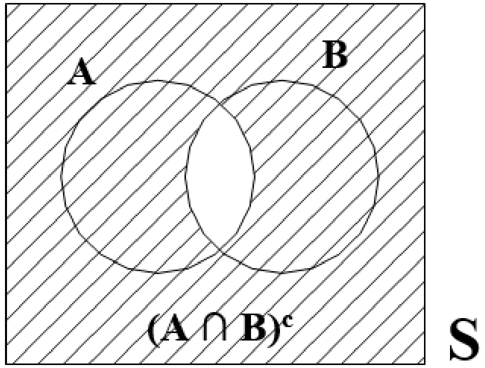
$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

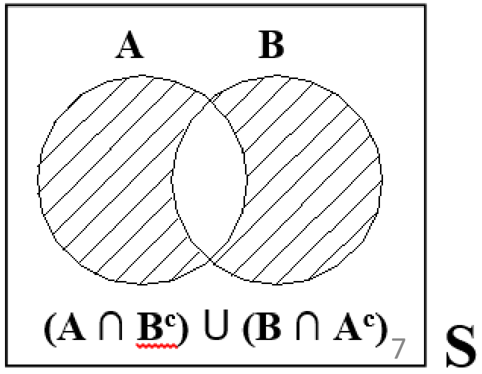












Example on a discrete sample space

An experiment has its sample space specified as: **S = {1, 2, 3, ……, 48, 49, 50}.** Define the events

- A : set of numbers divisible by 6
- B : set of elements divisible by 8
- C : set of numbers which satisfy the relation 2^n , n = 1, 2, 3,…
- Find: 1- A, B, C 2- A U B U C 3- A ∩ B ∩ C

- **The sample space S consists of discrete elements that are countable.**
- **Events defined on S consist of countable elements as well.**

Solution:

1. Events A, B, and C are:

- A = {6, 12, 18, 24, 30, 36, 42, 48}
- B = {8, 16, 24, 32, 40, 48}
- C = {2, 4, 8, 16, 32}
- 2- A U B U C = {6, 12, 18, 24, 30, 36, 42, 48, 8, 16, 32, 40, 2, 4}
- 3- A ∩ B ∩ C = { Ø }

Example on a continuous sample space

The sample space of an experiment is:

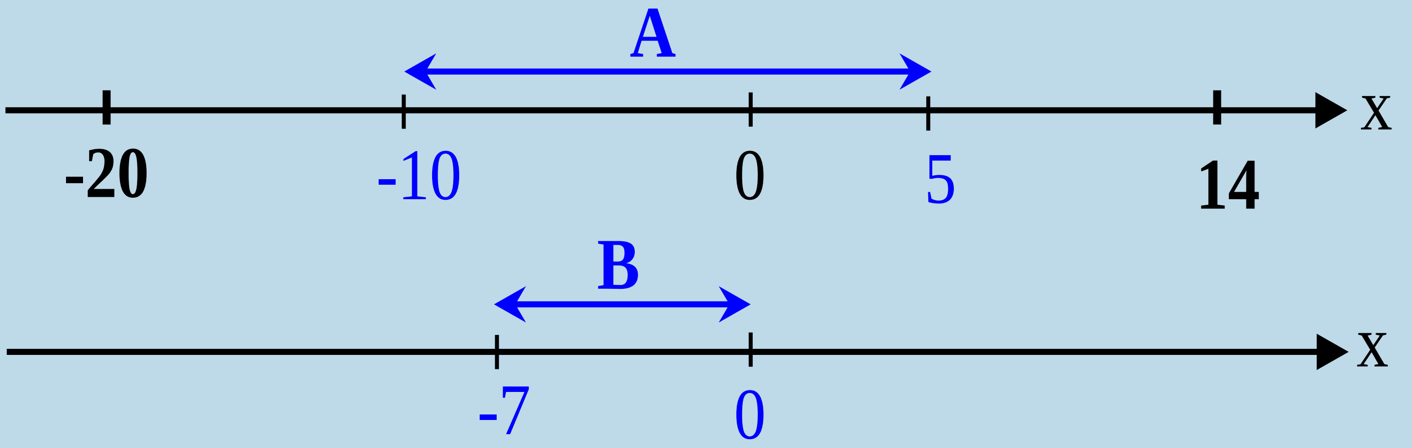
$S = \{-20 \leq x \leq 14\}$. If $A = \{-10 \leq x \leq 5\}$ and $B = \{-7 \leq x \leq 0\}$ find.

1- $A \cup B$ 2- $A \cap B$

Solution:

1- $A \cup B = \{-10 \leq x \leq 5\}$

2- $A \cap B = \{-7 \leq x \leq 0\}$



- **The sample space is uncountable and infinite in number.**
- **Here, the sample space is an interval on the real line.**
- **Events defined on S are intervals on the real line.**

Definitions of Probability

Four definitions of probability have evolved over the years

- Definition I: Classical (a priori)
- Definition II: Relative Frequency (a posteriori)
- Definition III: Subjective
- Definition IV: Axiomatic

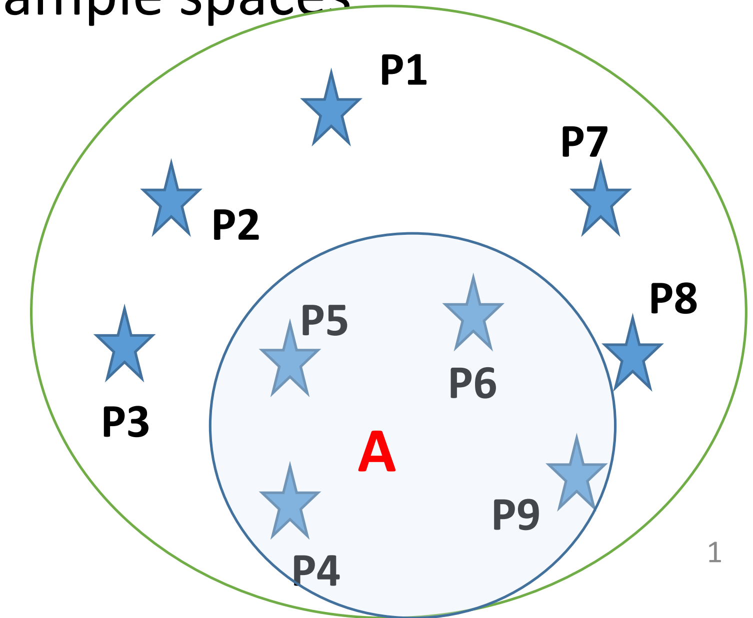
- **Classical Definition of Probability**: If the sample space S of an experiment consists of **finitely many outcomes** (points) that are **equally likely**, then the probability of event A, P(A), defined on S is:

$$P(A) = \frac{\text{Number of outcomes in A}}{\text{Number of outcomes in S}}$$

- **Thus, when A = S, P(S) = 1.**

Drawbacks of this definition

1. Limited to discrete and finite sample spaces
2. Does not handle discrete and countably infinite sample spaces
3. What happens when outcomes are not equally likely?
4. Does not apply to continuous sample spaces



Definitions of Probability

Example on the classical definition:

One integer is chosen at random from the set of numbers {1, 2, ……, 50}. What is the probability that the chosen number is divisible by 6? Assume all 50 outcomes are equally likely.

Solution

- S = {1, 2, 3, …………, 50}; Total number of points in S is 50.
- A = {6, 12, 18, 24, 30, 36, 42, 48}; Number of points in A is 8.
- Apply the classical definition to evaluate P(A). The answer

$$P(A) = \frac{\text{Number of elements in A}}{\text{Number of elements in S}} = \frac{8}{50}$$

Examples

Example : Consider the experiment of flipping a fair coin three times.

1	2	3
(H,T)	(H,T)	(H,T)

a. What is the sample space?

b. Define event: A = {The coin shows heads on all trials}.

c. Find P(A)

Solution:

a- Sample Space: S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}; 8 possible outcomes

b- A = {HHH}; 1 element. Hence **P(A) = 1/8**

Example: A fair die and a fair coin are flipped simultaneously.

1	2	3	4	5	6

(H,T)

- Find the sample space
- Find the event A= {number divisible by 2, Head}
- Find P(A)

Solution:

- a. Sample Space S={(1, H), (2, H), (3, H), (4, H), (5, H), (6, H), (1, T), (2, T), (3, T), (4, T), (5, T), (6, T)}; 12 pair of elements.
- A = {(2, H), (4, H), (6, H)}; 3 pairs of elements. Hence, **P(A) = 3/12**

Relative Frequency Definition

- Let an experiment be repeated N times under identical conditions. The relative frequency definition is given as:

$$P(A) = \lim_{N \to \infty} \frac{N_A}{N} = \frac{\text{Number of times event A occurs}}{\text{Number of times experiment is performed}}$$

- Clearly $0 \leq P(A) \leq 1$

0	0	0	1	0	0	0	1	0	0

Experiment is repeated N times

Problems with this definition:

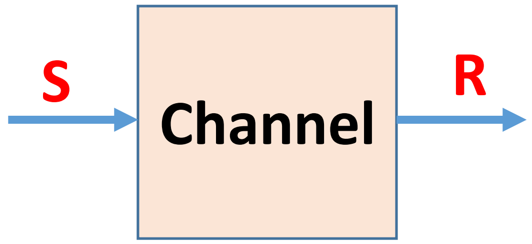
- If A does not occur in the sequence of trials (recall that this is a random experiment), then P(A) = 0
- If (A) occurs on each one of the N trials, then P(A) = 1
- How do you maintain identical conditions when repeating the experimet a large number of times?
- How much large should N be?

Relative Frequency Definition

Example:

In digital data transmission, the bit error probability has an unknown value p. If 10,000 bits are transmitted over a noisy communication channel and 5 bits were found to be in error, find the bit error probability (p).

- **Solution**
- The number of times the experiment is repeated is 10,000.
- The number of times event A appears is 5
- Hence, according to the relative frequency definition: **P(A) = 5/10,000**



A test frame of 10,000 bits

Sent Sequence **S**	**0**	**1**	**1**	**0**	**0**	**1**	**0**	**1**	**0**	**0**
Received Sequence **R**	**0**	**1**	**1**	**1**	**0**	**1**	**0**	**0**	**0**	**0**

Subjective Definition of Probability

- Probability is defined as a person's measure of belief that some given event will occur.
- ***Example:***
- What is the probability of developing a safe vaccine to the coronavirus by September 2020?
- Any number we might come up with, would be our own personal (subjective) assessment of the situation.
- **Example:**
- What is the probability that your grade in this course will be 90 or above?

Axiomatic Definition of Probability

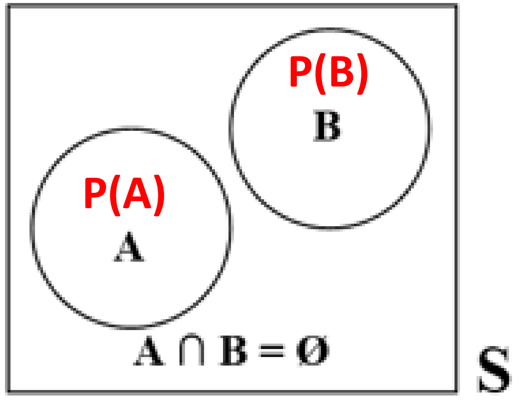
- Given a sample space S. For each event A on S (subset of S), there is an associated number P(A), called the probability of (A), such that the following axioms of probability are satisfied:
- **Axiom 1**: $P(A \geq 0)$; Probability is nonnegative. It can take the value 0 but never negative
- **Axiom 2**: $P(S = 1)$; Probability of the sample space is a certain event.
- **Axiom 3**: For two mutually exclusive events A and B, i.e., ($A \cap B = \emptyset$)

  $P(A \cup B) = P(A) + P(B)$; ($A \cap B = \emptyset$)
- If S has infinitely many points, axiom (3) is to be replaced by:

  $P(A_1 \cup A_2 \cup A_3 \cup \ldots\ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$

  where $A_1, A_2, A_3 \ldots\ldots$ are mutually exclusive events

  ($A_1 \cap A_2 = \emptyset$, $A_1 \cap A_3 = \emptyset$, $A_2 \cap A_3 = \emptyset$, …)



Basic Theorems for Probability

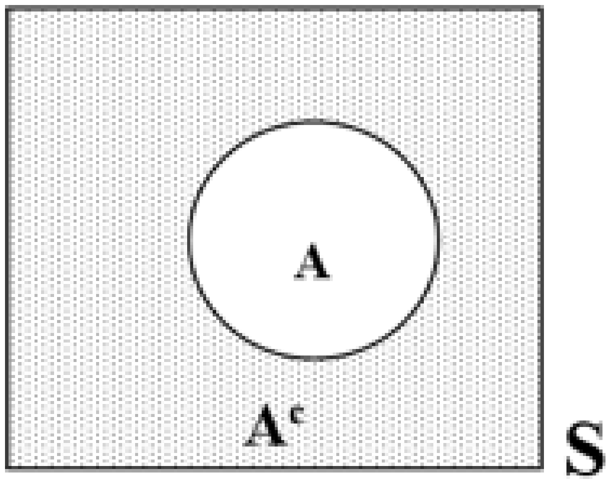
1. $P(A^c) = 1 – P(A)$

Proof:

$S = A \cup A^c$

$P(S) = P(A) + P(A^c)$; mutually exclusive events

$1 = P(A) + P(A^c) \quad \rightarrow \quad P(A^c) = 1 - P(A)$



2- $P(\emptyset) = 0$

Proof:

$S = S \cup S^c$

$S = S \cup \emptyset \qquad ; S^c = \emptyset$

$P(S) = P(S) + P(\emptyset) \quad \rightarrow \quad P(\emptyset) = 0$

Basic Theorems for Probability

3-$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof:

For events (A) and (B) in a sample space:

$\{A \cup B\} = \{A \cap B^c\} \cup \{A \cap B\} \cup \{B \cap A^c\}$; **union of 3 disjoint events**

$\{A \cup B\} =$ **1** U **2** U **3**

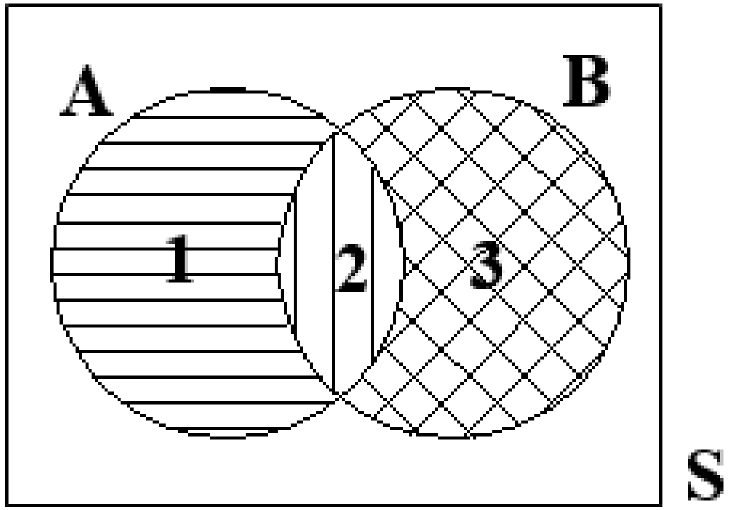
Where events (1) and (2) and (3) are all mutually exclusive

$P(A \cup B) = P(1) + P(2) + P(3)$

$P(A) = P(1) + P(2)$; $P(B) = P(2) + P(3)$

➔ $P(A \cup B) = \{P(1) + P(2)\} + \{P(2) + P(3)\} - \{P(2)\}$; **Add and subtract P(2)**

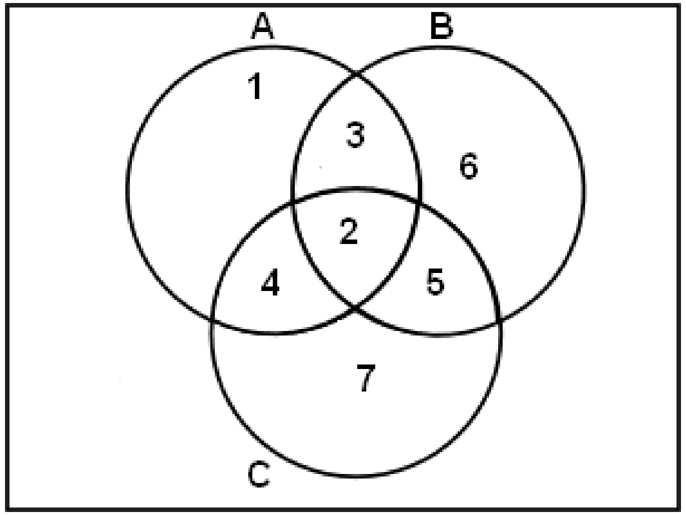
➔ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



3. **$P(A \cup B \cup C) = P(A) + P(B) + P(C)$**

$- P(A \cap B) - P(A \cap C) - P(B \cap C)$

$+ P(A \cap B \cap C)$

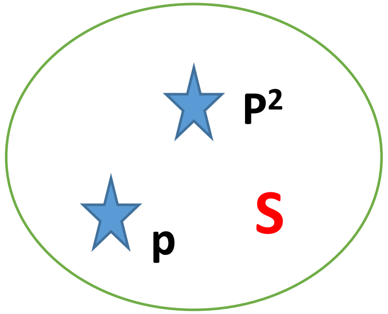


Sketch of the Proof

You can partition the union into 7 disjoint events, as shown in the figure, and use steps similar to those used to find the union of two events

Examples

Example: An experiment has two possible outcomes; the first occurs with probability (P), the second with probability (P^2). Find (P).



Solution: Apply the axioms of probability

- P(S) = 1; Axiom 3
- $P + P^2 = 1$; Axiom 2
- $P^2 + P - 1 = 0$
- $P = \dfrac{-1+\sqrt{5}}{2}$; Only the positive root is taken, by virtue of Axiom 1

$$P = \frac{-1-\sqrt{5}}{2}$$

This is a second root of the quadratic equation, however it violates the axioms of probability (**probability is non-negative**).

Examples

Example: One integer is chosen at random from the set of numbers {1, 2, ……, 50}. What is the probability that the chosen number is divisible by 6 given that the probability of occurrence of an even number is twice as likely as that of an odd number?

Solution: Let (P) be the probability of occurrence of an odd number, then (2P) will be the probability of occurrence of an even number.

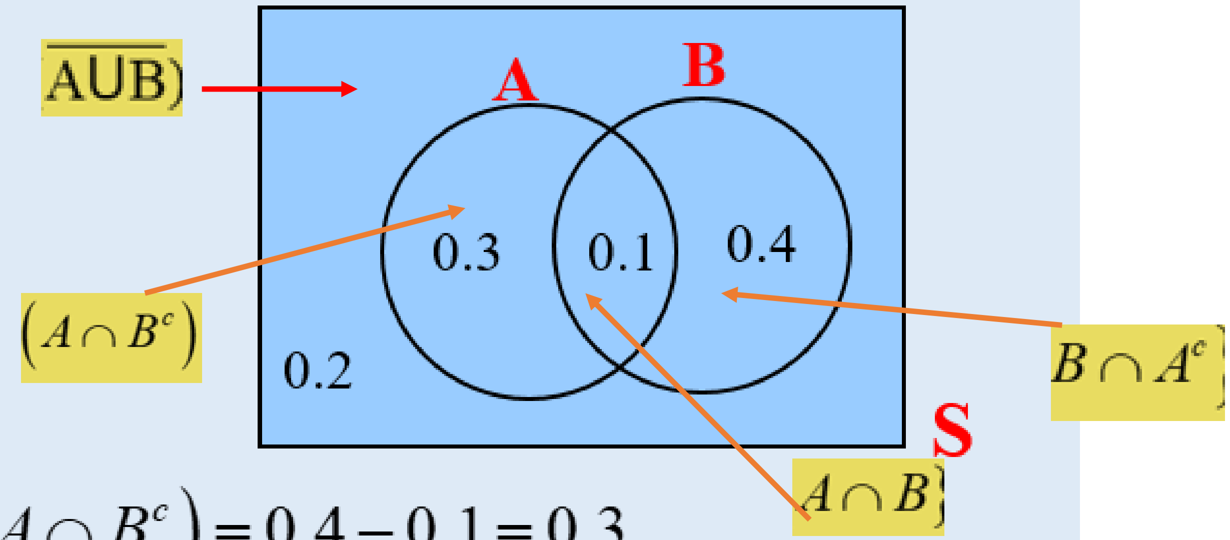
- $P(S) = P(even) + P(odd) = 1$;
- $(25)(2P) + (25)(P) = 1$
- $(50 + 25)(P) = 1$
- Hence **P=1/75**
- $A = \{6, 12, 18, 24, 30, 36, 42, 48\}$
- $P(A) = (8)(2P) = 16/75$

1	**2**	**3**	**4**	**5**	**6**	**.**	**.**	**49**	**50**
p	**2p**	**p**	**2p**	**p**	**2p**	**.**	**.**	**p**	**2p**

EXAMPLE: Let (A) and (B) be any two events defined on (S). Suppose that P(A) = 0.4, P(B) = 0.5, and P(A ∩ B) = 0.1.

Find the probability that:

1- Events A or B but not both occur.
2- None of the events A or B will occur.
3- At least one event will occur.
4- Both events occur.



SOLUTION:

$$P(A) = P\left(\left\{A \cap B^c\right\} \cup \left\{A \cap B\right\}\right) \Rightarrow P\left(A \cap B^c\right) = 0.4 - 0.1 = 0.3$$

$$P(\mathrm{B}) = P\left(\left\{B \cap A^c\right\} \cup \left\{A \cap B\right\}\right) \Rightarrow P\left(B \cap A^c\right) = 0.5 - 0.1 = 0.4$$

P(at least one event) = P(AU B)
P(Both) = P(A∩)
P(None) = P(complement of P(AU B)

1- P(A or B but not both) $= P\left(A \cap B^c\right) + P\left(B \cap A^c\right) = 0.3 + 0.4 = 0.7$

Note that:

P(A U B) = P(A) + P(B) – P(A ∩ B)
P(A U B) = 0.4 + 0.5 – 0.1 = 0.8

Let X be number of events occurring
P(S)=P(X=0)+P(X=1)+P(X=2)
P(S)=P(no events)+P(exactly one event)+P(both events)

2- P(none of the events will occur) = P($\overline{AUB}$) = 0.2
3- P(at least one event will occur) = P(A or B or both) P(A U B) = 0.8
4- P(both events occur) = P(A ∩ B) = 0.1

Discrete and Continuous Probability Functions

Discrete Sample Space and Discrete Probability Function

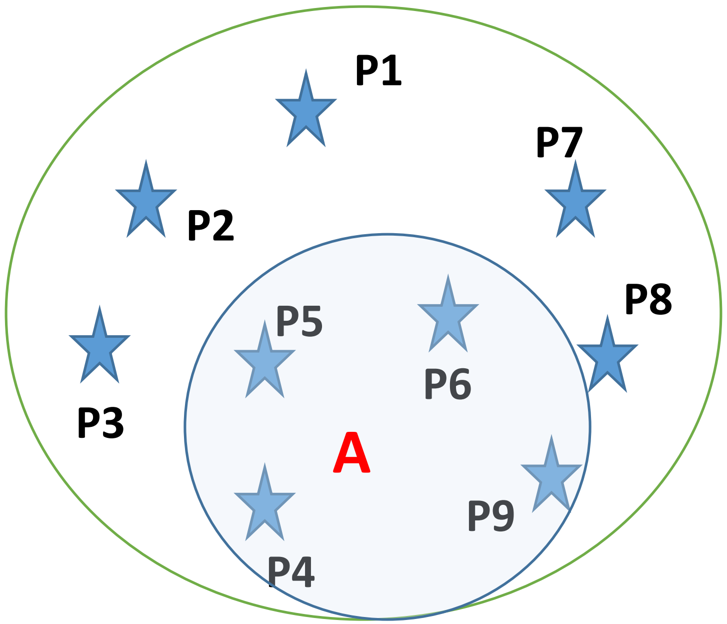
- If the sample space generated by an experiment contains either a countable *finite* or a *countable infinite* number of outcomes, then it is called a *discrete* sample space.
- Any probability assignment on that space such that:

  a. $P(s_i) \geq 0$

  b. $\sum_{s_i \in S} P(s_i) = 1$

  Axioms of probability applied to discrete sample spaces

  is said to be a discrete probability function.
- If (A) is an event defined on (S), then $P(A) = \sum_{s_i \in A} P(s_i)$



- For example, the sample space, S = {1 , 2 , 3 , 4 , 5 , 6} is countably finite.
- The set of positive integers, S = {1 , 2 , 3 , … } is countably infinite

Discrete Sample Space and Discrete Probability Function

Example on a finitely countable sample space: A sample space "S" consists of the integers 1 to 6 inclusive. **The probability of each outcome is proportional to its magnitude**. If one integer is chosen at random, what is the probability that an even integer appears?

Solution:

- Sample Space "S" = {1 , 2 , 3 , 4 , 5 , 6}; finitely countable sample space
- Event (A) = {2 , 4 , 6}

$$\sum_{i=1}^{n} i = \frac{\mathrm{n(n+1)}}{2}; \text{ arithmatic series}$$

$$P(S) = 1 = \sum_{i=1}^{6} p(i) = \sum_{i=1}^{6} i\alpha = \frac{6(6+1)}{2}\alpha = 1; \text{ arithmatic series}$$

1	2	3	4	5	6
α	2α	3α	4α	5α	6α

$$\text{Hence the proportionality constant } \alpha = \frac{1}{21}$$

$$P(A) = P(2) + P(4) + P(6) = \frac{2}{21} + \frac{4}{21} + \frac{6}{21} = \frac{12}{21}$$

Discrete Sample Space and Discrete Probability Function

Example on an infinitely countable sample space: The discrete probability function for the countably infinite sample space S = {X: 1, 2, 3, …} is:

$$P(x) = \frac{C}{x^2}; \quad x = 1, 2, 3, \ldots$$

a- Find the constant "C" so that P(x) is valid discrete probability function.

b- Find the probability that the outcome of the experiment is a number less than 4.

SOLUTION:

a. By Axiom 2, P(S) = 1

$$\sum_{x=1}^{\infty} \frac{C}{x^2} = 1 \Rightarrow \quad C\sum_{x=1}^{\infty} \frac{1}{x^2} = 1 \quad \Rightarrow C\frac{\pi^2}{6} = 1 \quad \Rightarrow C = \frac{6}{\pi^2}$$

Basel problem, first posed by Pietro Mengoli in 1644 and solved by Leonhard Euler in 1734.

b.

$$P(A) = P(1) + P(2) + P(3)$$

$$P(A) = \frac{6}{\pi^2}\left(\frac{1}{(1)^2} + \frac{1}{(2)^2} + \frac{1}{(3)^2}\right) = \frac{49}{6\pi^2}$$

Discrete and Continuous Probability Functions

Continuous Sample Space and Continuous Probability Function

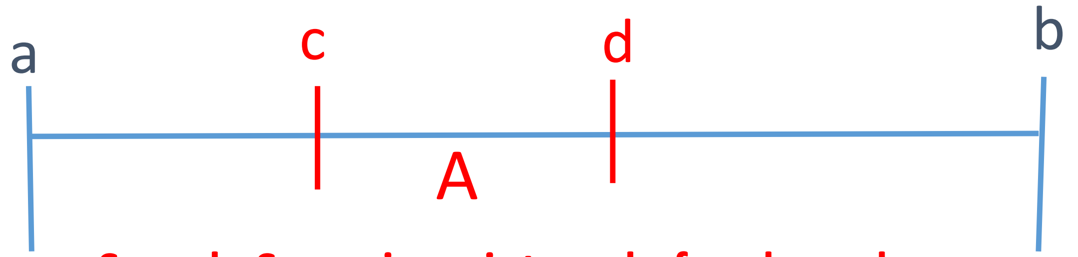
If the sample space associated with an experiment is an interval of real numbers, then S has an *uncountable infinite number* of points and S is said to be *continuous*.

- Let $f(x)$ be a real-valued function defined on S such that:

$$a.\ f(x) \geq 0$$

$$b.\ \int_{\text{All x}} f(x)\,\text{dx} = 1$$

Axioms of probability applied to continuous sample spaces



Sample Space is an interval of real numbers

- The function $f(x)$ that satisfies these conditions is called a continuous probability function (also known as the probability density function)
- If (A) is an event defined on S, then

$$\text{P(A)} = \int_{\text{x} \in \text{A}} f(x)\,\text{dx}$$

Continuous Sample Space and Continuous Probability Function

Example: Let the sample space of an experiment be: $S = \{1 \le x \le 2\}$. The continuous probability function defined over S

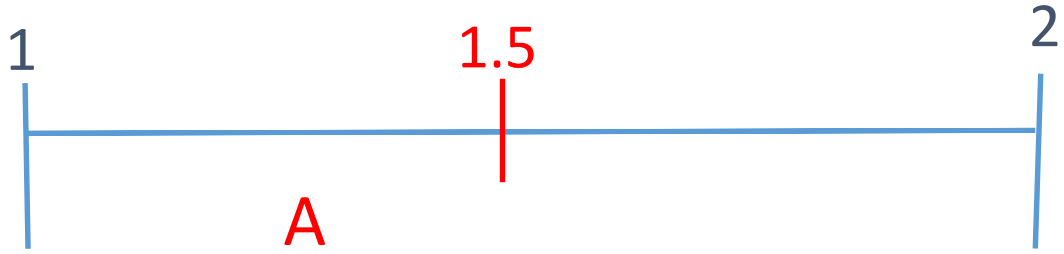
$$f(x) = \frac{k}{x^2} =, \ 1 \le \text{x} \le 2$$

a. Find k so that $f(x)$ is a valid continuous probability function.
b. Find $\mathrm{P}(1 \le x \le 1.5)$

- **Solution**:

$$a. P(S) = \int_1^2 f(x)\,dx = 1 \Rightarrow \int_1^2 \frac{k}{x^2}\,dx = 1 \Rightarrow k = 2$$

$$b. P(x \le 1.5) = \int_1^{1.5} \frac{k}{x^2}\,dx = \frac{2}{3}$$

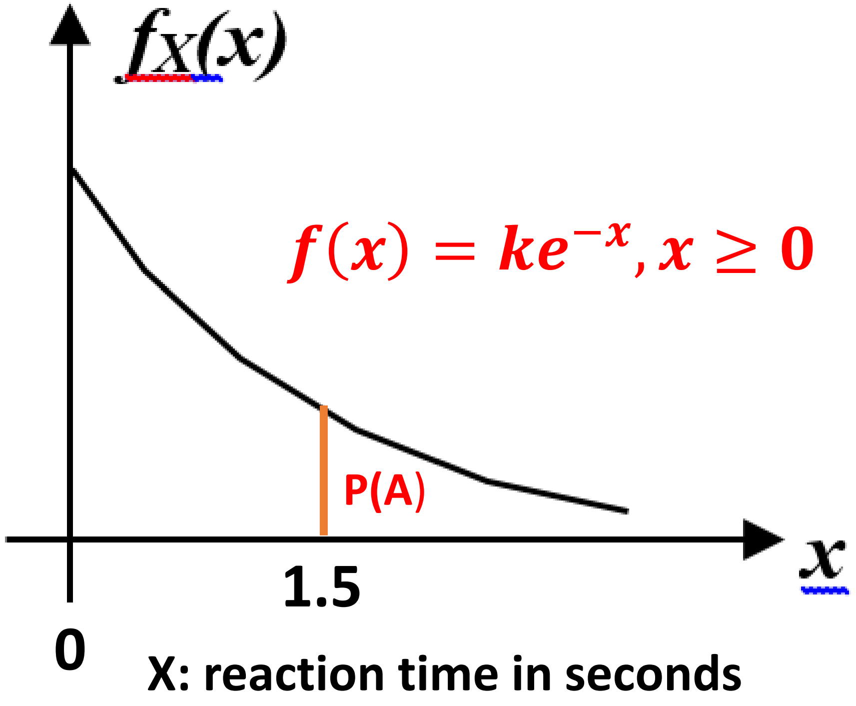


Continuous Sample Space and Continuous Probability Function

- **Example**: Assume that the reaction time, X, of a driver over the age of 70 to a certain visual stimulus is described by a continuous probability function of the form $f(x) = ke^{-x}, x \geq 0$. where x is measured in seconds. Let A be the event "Driver requires longer than 1.5 seconds to react". Find P(A).

- **Solution:**
- First, we have to find k.
- $\int_0^\infty ke^{-x}\,dx = 1 \Rightarrow \boldsymbol{k = 1}$.
- The probability of A is calculated as:
- $P(A) = P(X \geq 1.5)$

$$= \int_{1.5}^\infty e^{-x}\,dx = e^{-1.5}$$



EXAMPLE: The outcome of an experiment is either a success with probability p or a failure with probability (1-p). If the experiment is to be repeated until a success comes up for the first time. Let X be the number of times the experiment is performed, then the discrete probability function for the countably infinite sample space is

$$P(x) = p(1-p)^{x-1}\ ;\ x = 1, 2, \ldots$$

Geometric Distribution

What is the probability that a success occurs on an odd-numbered trial?

P	1- P
S	F

SOLUTION: The sample space for the experiment is S={1, 2, 3, ….}

Let A be the event that a success occurs on an odd numbered trial, then A consists of the sample points: A = {1, 3, 5, …}

$$1 + u + u^2 + u^3 + \ldots = \frac{1}{1-u}$$

$$\mathrm{P(A)} = \mathrm{P(1)} + \mathrm{P(3)} + \mathrm{P(5)} + \ldots$$

$$P(A) = p(1-p)^{1-1} + p(1-p)^{3-1} + p(1-p)^{5-1} + \ldots$$

$$P(A) = p(1 + q^2 + q^4 + \ldots) = \boxed{\frac{p}{1-q^2}};\quad q = 1-p$$

, by virtue of the geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x};\quad |x| < 1;\ \text{Geometric Series}$$

X=1	S		
X=2	F	S	
X=3	F	F	S

In the special case when p = ½, P(A) becomes

$$\mathrm{P(A)} = \frac{0.5}{1-(0.5)^2} = \frac{2}{3}$$

Conditional Probability

Motivating Example: Consider the experiment of rolling a fair die once.

- What is the probability that number 4 appears?
  - Here, S = {1, 2, 3, 4, 5, 6}; **equally likely**
  - Event A = {4}
  - Using the classical definition of probability, P(A) = 1/6.

1	2	3	4	5	6	S

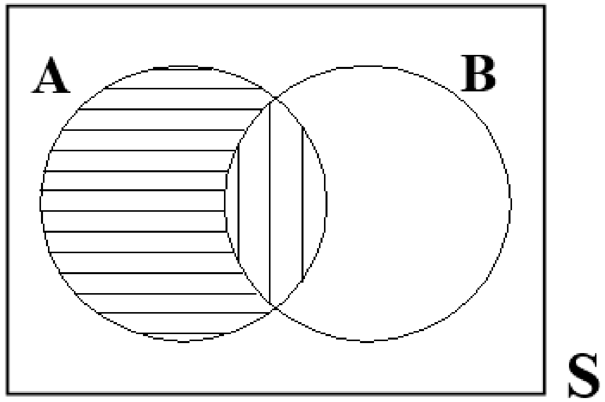
- Now, suppose that someone told you that the outcome of the experiment is an even number. We ask the same question: What is the probability that number 4 appears?
  - B = {2, 4, 6}; the additional or "conditional probability", **outcome is even**
  - Event A = {4}
  - **P(A/B) = 1/3; P(number 4/outcome is even)**

2	4	6	B

- Note that event A is chosen relative to the reduced sample space B.

Conditional Probability

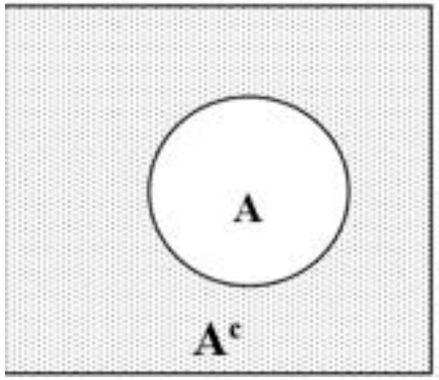
- Given two events (A) and (B) with P(A) > 0 and P(B) > 0. We define the Conditional Probability of (A) given (B) has occurred as:



$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad (1)$$

- Here, B serves as a new (reduced) sample space
- Also, P(A/B) is the fraction of P(A ∩ B) relative to P(B).
- In a similar way, the probability of (B) given (A) has occurred is defined as

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad (2)$$



- From (1), we get: $P(A \cap B) = P(B)P(A|B)$
- From (2), we get: $P(A \cap B) = P(A)P(B|A)$
- Finally, $P(A \cap B) = \boldsymbol{P(A)P(B|A) = P(B)P(A|B)}$

$$\boldsymbol{P(A) = \frac{P(A)}{1} = \frac{P(A)}{P(S)}}$$

Conditional Probability

Motivating Example Revisited: Consider the experiment of rolling a fair die once.

- Find the probability that number 4 appears given that the experiment outcome is an even number.
- **Solution:** This example is solved using the definition of conditional probability
- $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- Here, S = {1, 2, 3, 4, 5, 6}
- Event A = {4}
- Event B = {2, 4, 6}; the condition;
- $A \cap B = \{4\}; \qquad \Rightarrow \mathrm{P}(A \cap B) = 1|6.$
- P(B) = 1/6 + 1/6 + 1/6 = 3/6
- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1|6}{3|6} \Rightarrow \quad \boldsymbol{P(A|B) = \frac{1}{3}};$ **same as before**

Conditional Probability

Example on a finitely countable sample space: A sample space "S" consists of the integers 1 to 6 inclusive. Each outcome has an associated probability proportional to its magnitude. If one number is chosen at random,

a. What is the probability that number 1 appears?

b. What is the probability that number 1 appears given that the outcome of the experiment is an odd number?

Solution:

- Sample Space "S" = {1 , 2 , 3 , 4 , 5 , 6}; finitely countable sample space
- Event (A) = {1}
- B = {1, 3, 5}
- $\{A \cap B\} = \{\mathbf{1}\}$

1	**2**	**3**	**4**	**5**	**6**
α	**2α**	**3α**	**4α**	**5α**	**6α**

$$P(S) = 1 = \sum_{i=1}^{6} p(i) = \sum_{i=1}^{6} \alpha(i) = \frac{6(6+1)}{2}\alpha = 1$$

Hence the proportionality constant $\alpha = \dfrac{1}{21}$

$$\Rightarrow P(A) = P(1) = \frac{1}{21}; \text{ unconditional prob.}$$

$$P(\mathrm{A} \mid \mathrm{B}) = \frac{P(A \cap B)}{P(B)} == \frac{P(1)}{\mathrm{P}(1) + P(3) + P(5)}$$

$$P(\mathrm{A} \mid \mathrm{B}) = \frac{(1/21)}{(1/21) + (3/21) + (5/21)} = \frac{1}{9}$$

Conditional Probability

Theorem: Multiplication Rule

If (A) and (B) are events in a sample space (S) and P(A) ≠ 0, P(B) ≠ 0, then:

$$P(A \cap B) = P(A)\,P(B/A) = P(B)\,P(A/B)$$

EXAMPLE: A certain computer becomes inoperable if two components A and B both fail. The probability that A fails is 0.001 and the probability that B fails is 0.005. However, the probability that B fails increases by a factor of 4 if A has failed. Calculate the probability that:

a- The computer becomes inoperable.
b- A will fail if B has failed.

SOLUTION: $P(A) = 0.001$; $P(B) = 0.005$ $P(B/A) = 4 \times 0.005 = 0.020$

a- The system fails when both A and B fail, i.e.,

$$P(A \cap B) = P(A)\,P(B/A)$$

$$P(A \cap B) = 0.001 \times 0.020 = 0.00002$$

b- $P(A \cap B) = P(A)\,P(B/A) = P(B)\,P(A/B) \Rightarrow P(A/B) = \dfrac{0.001 \times 0.020}{0.005} = 0.004$

Theorem: Multiplication Rule

If A, B and C are events in a sample space (S) and P(A) ≠ 0, P(B) ≠ 0, P(C) ≠ 0 then:

$$\mathbf{P(A \cap B \cap C) = P(A)\, P(B/A)\, P(C/B,A)}$$

EXAMPLE: A box contains 20 non-defective (N) items and 5 defective (D) items. Three items are drawn without replacement.

a. Find the probability that the sequence of objects obtained is (NND) in the given order.
b. Find the probability that exactly one defective item is obtained.

SOLUTION:

a. $P(N \cap N \cap D) = P(N) \times P(N/N) \times P(D/N,N)$

$$P(NND) = \left(\frac{20}{25}\right)\left(\frac{20-1}{25-1}\right)\left(\frac{5}{25-2}\right) = \left(\frac{20}{25}\right)\left(\frac{19}{24}\right)\left(\frac{5}{23}\right)$$

b. One defective item is obtained, when any one of the following sequences occurs:

(NND), (NDN), (DNN)

The probability of getting one defective item is the sum of the probabilities of these sequences and is given as:

$$\left(\frac{20}{25}\right)\left(\frac{19}{24}\right)\left(\frac{5}{23}\right) + \left(\frac{20}{25}\right)\left(\frac{5}{24}\right)\left(\frac{19}{23}\right) + \left(\frac{5}{25}\right)\left(\frac{20}{24}\right)\left(\frac{19}{23}\right) = 3\left(\frac{20}{25}\right)\left(\frac{19}{24}\right)\left(\frac{5}{23}\right)$$

Later in Chapter 2, we will see that Part (b) can be solved using the *hyper-geometric distribution.*

N = 20
D = 5
25

N = 19
D = 5
24
N

ND=18
D=5
23
N
N

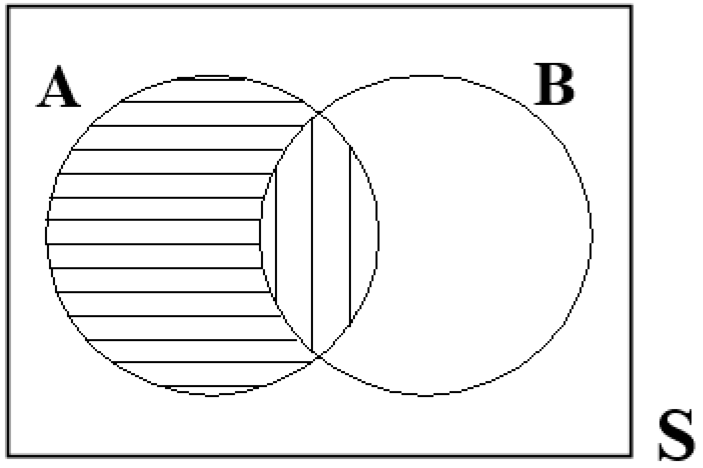
Statistical Independence

- **REVIEW:** Given two events (A) and (B) with P(A) > 0 and P(B) > 0. We define the Conditional Probability of (A) given (B) has occurred as:

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad (1)$$

- Here, B serves as a new (reduced) sample space
- Also, P(A/B) is the fraction of P(A ∩ B) relative to P(B).
- In a similar way, the probability of (B) given (A) has occurred is defined as

$$P(B|A) = \frac{P(A\cap B)}{P(A)} \quad (2)$$



- From (1), we get: $P(A \cap B) = P(B)P(A|B)$
- From (2), we get: $P(A \cap B) = P(A)P(B|A)$
- Finally,
- $\boldsymbol{P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)}$

Statistical Independence

Definition: Statistical Independence

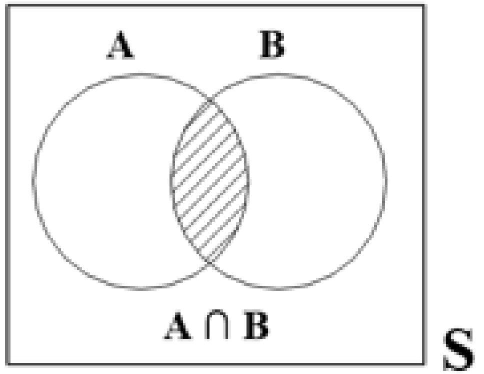
Two events (A) and (B) are said to be statistically independent if $P(A \cap B) = P(A)\,P(B)$

From this definition, we conclude that:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\,P(B)}{P(B)} = P(A) \Rightarrow \text{A posteriori} = \text{A priori probability}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)\,P(B)}{P(A)} = P(B)$$

This means that the probability of A does not depend on the occurrence or nonoccurrence of B and vice versa. Hence, the given information does not change our initial perception about the two given probabilities.



Independence of Three Events

Events (A), (B) and (C) are independent if the following conditions are satisfied:

- $P(A \cap B) = P(A)\,P(B)$
- $P(A \cap C) = P(A)\,P(C)$
- $P(B \cap C) = P(B)\,P(C)$
- $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

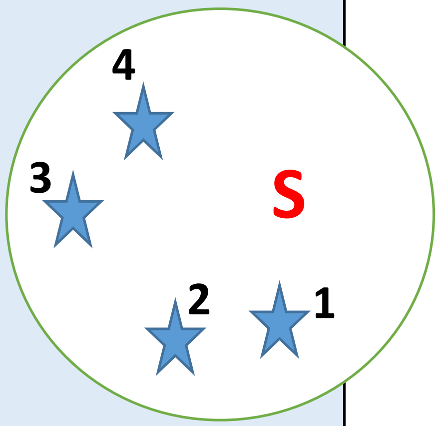
P(intersection) = product of probabilities

Example: Independence of Two Events

- The sample space of an experiments consists of the integers S ={1, 2, …,50}. All points in the sample space are equally likely. Define the two events A and B as:
- {A}: Integers divisible by 6
- {B}: Integers divisible by 8.
- Are A and B independent?
- **Solution:** For A and B to be independent, the following condition must hold
- $\boldsymbol{P(A \cap B) = P(A)P(B)}$ ….(1)
- A = {6, 12, 18, 24, 30, 36, 42, 48}, ⇒ $\boldsymbol{P(A) = 8/50}$. Equally likely outcomes
- B = {8, 16, 24, 32, 40, 48} ⇒ $\boldsymbol{P(B) = 6/50}$
- $A \cap B = \{24, 48\} \Rightarrow \boldsymbol{P(A \cap B) = 2/50}$
- Therefore, $P(A \cap B) \neq P(A)P(B)\ since\ \frac{2}{50} \neq \frac{8}{50} * \frac{6}{50}$

Example: Independence of three events

EXAMPLE Consider an experiment in which the sample space contains four outcomes $\{S_1, S_2, S_3, S_4\}$ such that $P(S_i) = \frac{1}{4}$. Let events (A), (B) and (C) be defined as:



$A = \{S_1, S_2\}$, $B = \{S_1, S_3\}$, $C = \{S_1, S_4\}$

Are these events independent?

SOLUTION: $P(A) = P(B) = P(C) = \frac{1}{2}$

$(A \cap B) = \{S_1\}$; $(A \cap C) = \{S_1\}$; $(B \cap C) = \{S_1\}$; $(A \cap B \cap C) = \{S_1\}$

$P(A \cap B) = \frac{1}{4}$; $P(A \cap C) = \frac{1}{4}$; $P(B \cap C) = \frac{1}{4}$; $P(A \cap B \cap C) = \frac{1}{4}$

Check the conditions:

$P(A \cap B) = \frac{1}{4} = P(A)\,P(B) = \frac{1}{2} \times \frac{1}{2}$; $P(A \cap C) = \frac{1}{4} = P(A)\,P(C) = \frac{1}{2} \times \frac{1}{2}$

$P(B \cap C) = \frac{1}{4} = P(B)\,P(C) = \frac{1}{2} \times \frac{1}{2}$

$P(A \cap B \cap C) = \frac{1}{4} \neq P(A)\,P(B)\,P(C) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

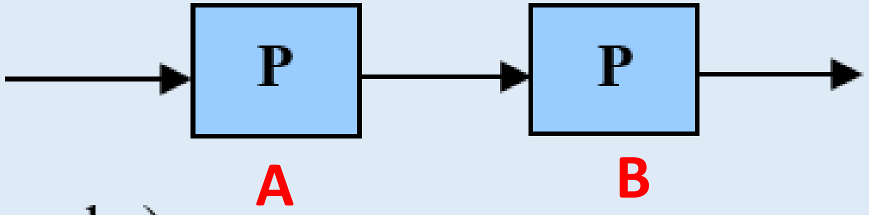
- $P(A \cap B) = P(A)\,P(B)$
- $P(A \cap C) = P(A)\,P(C)$
- $P(B \cap C) = P(B)\,P(C)$
- $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

➔ **Events are not independent (even though they are pair-wise independent)**

Example: Reliability of a Series System

EXAMPLE: Reliability of a series system

Suppose that a system is made up of two components connected in series, each component has a probability (P) of working "**Reliability**". What is the probability that the system works assuming that components work independently?



SOLUTION:

P(system works) = P($\overset{A}{\text{component 1 works}} \cap \overset{B}{\text{component 2 works}}$)

P(system works) = $P \times P = P^2$

**** The probability that the system works is also known as the "Reliability" of the system.***

$$P(A \cap B) = P(A)P(B)$$

Example: Examples: Reliability of a Parallel System

EXAMPLE: Reliability of a parallel system

Suppose that a system is made up of two components connected in parallel. The system works if at least one component works properly. If each component has a probability (P) of working "**Reliability**" and components work independently, find the probability that the system works.

SOLUTION:

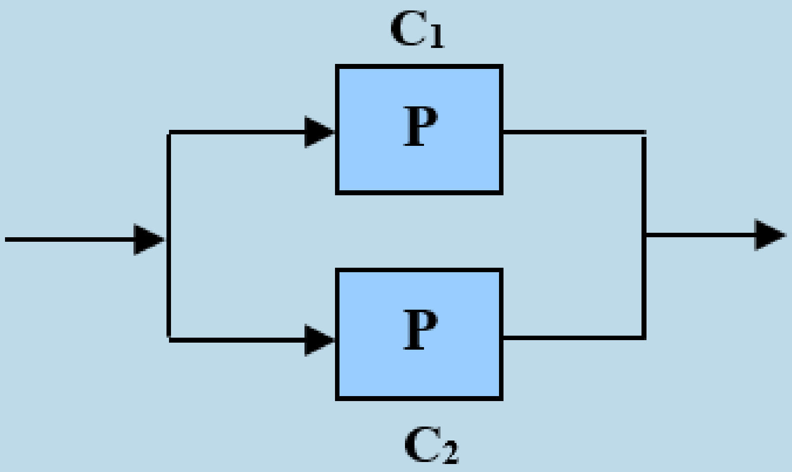
Reliability of the system = P(system works)

P(system works) $= P(C_1 \text{ or } C_2 \text{ or both } C_1 \text{ and } C_2 \text{ works})$

$= P(C_1 \cup C_2)$ Probability of at least one component works

$= P(C_1) + P(C_2) - P(C_1 \cap C_2)$

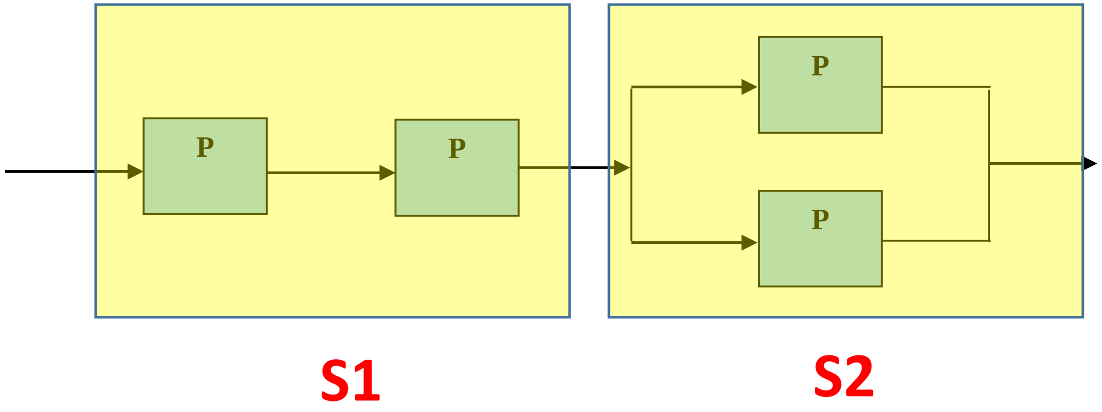
$= P + P - (P \times P) = 2P - P^2$



** This system fails if both components fail.*

Example: Reliability of a Mixed System

- Find the reliability of the shown mixed system, assuming that all components work independently, and P is the reliability (probability of working) of each component.
- $\boldsymbol{P(System\ Works) = P(S_1 \cap S_2)}$; System are connected in **series**
- $\boldsymbol{P(System\ Works) = P(S_1)P(S_2)}$; S1 and S2 are independent
- In the previous two examples, we found the reliabilities of S1 and S2. Therefore,
- $\boldsymbol{P(System\ Works) = P^2(2P - P^2)}$



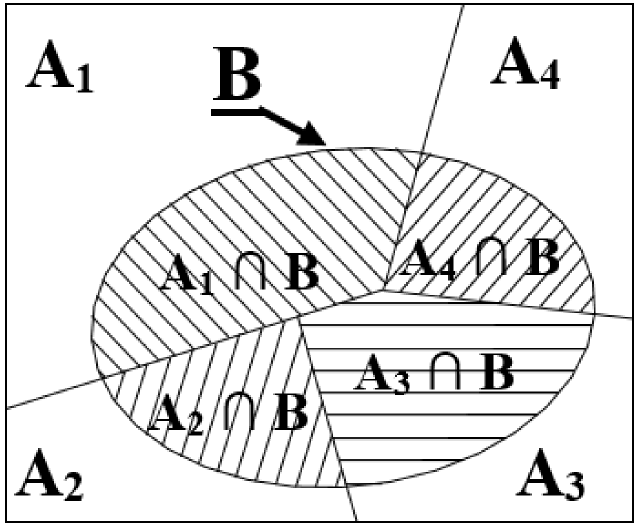
Theorem of Total Probability and Bayes' Theorem

Let $A_1, A_2, \ldots, A_n$ be a set of events defined over (S) such that:

$S = A_1 \cup A_2 \cup \ldots \cup A_n$; $A_i \cap A_j = \emptyset$ for $i \neq j$, and $P(Ai) > 0$ for $i = 1, 2, 3, \ldots n$.

For any event (B) defined on (S):

$$P(B) = P(A_1)\, P(B/A_1) + P(A_2)\, P(B/A_2) + \ldots\ldots + P(A_n)\, P(B/A_n)$$



Proof: For events (A) and (B) in a sample space:

$$B = \{A_1 \cap B\} \cup \{A_2 \cap B\} \cup \{A_3 \cap B\} \cup \{A_4 \cap B\}$$

Since these events are, disjoint, then:

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) + P(A_4 \cap B)$$

But $P(A \cap B) = P(A)\, P(B/A) = P(B)\, P(A/B)$

$$P(B) = P(A_1)\, P(B/A_1) + P(A_2)\, P(B/A_2) + P(A_3)\, P(B/A_3) + P(A_4)\, P(B/A_4)$$

Bayes' Theorem: If $A_1, A_2, A_3, \ldots, A_n$ are disjoint events defined on (S), and (B) is another event defined on (S) (same conditions as above), then:

$$P(A_j/B) = \frac{P(A_j)\, P(B/A_j)}{\sum_{i=1}^{n} P(A_i)\, P(B/A_i)} = \frac{P(A_j \cap B)}{P(B)}$$

From the previous two lectures

$P(A \cap B) = P(A)P(B|A)$; general

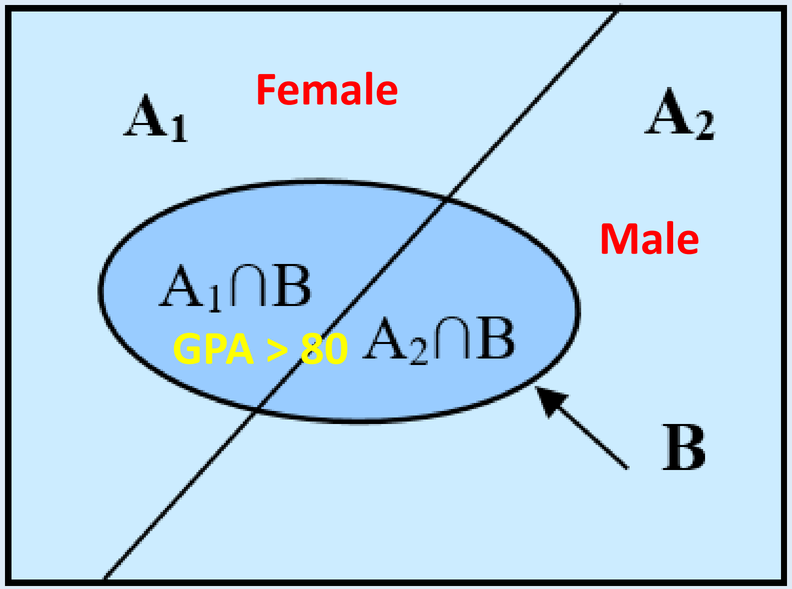
$P(A \cap B) = P(A)P(B)$; indep

$$P(A|C) = \frac{P(A \cap C)}{P(C)}$$

Theorem of Total Probability and Bayes' Theorem: Example

EXAMPLE: If female students constitute 30% of the student body in the Faculty of Engineering and 40% of them have A GPA > 80, while 25 % of the male students have their GPA > 80. What is the probability that a person selected at random will have a GPA > 80?

SOLUTION



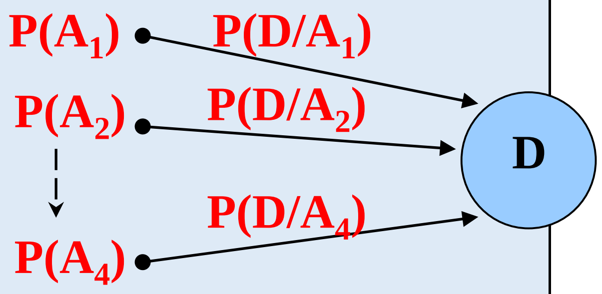
A_1 = Event representing the selected person is a female
A_2 = Event representing the selected person is a male
B = Event representing GPA > 80
$P(A_1) = 0.3$
$P(A_2) = 0.7$
$B = (A_1 \cap B) \cup (A_2 \cap B) \Rightarrow P(B) = P(A_1 \cap B) + P(A_2 \cap B)$
$P(B) = P(A_1)\, P(B/A_1) + P(A_2)\, P(B/A_2)$
$P(B) = (0.3 \times 0.4) + (0.7 \times 0.25)$
$P(B) = 0.295$

Theorem of Total Probability and Bayes' Theorem: Example

EXAMPLE: In a factory, four machines produce the same product. Machine A_1 produces 10% of the product, A_2 20%, A_3 30%, and A_4 40%. The proportion of defective items produced by the machines follows:

$P(D/A_1) = 0.001$; $P(D/A_2) = 0.005$; $P(D/A_3) = 0.005$; $P(D/A_4) = 0.002$

If an item selected at random is found to be defective, what is the probability that the item was produced by machine A_1?



SOLUTION: The proportion of items produced by the machines are

$P(A_1) = 0.1$; $P(A_2) = 0.2$; $P(A_3) = 0.3$; $P(A_4) = 0.4$

Let D be the event: Selected item is defective

$$P(D) = P(A_1)\,P(D/A_1) + P(A_2)\,P(D/A_2) + P(A_3)\,P(D/A_3) + P(A_4)\,P(D/A_4)$$

$$P(D) = (0.1 \times 0.001) + (0.2 \times 0.005) + (0.3 \times 0.005) + (0.4 \times 0.002)$$

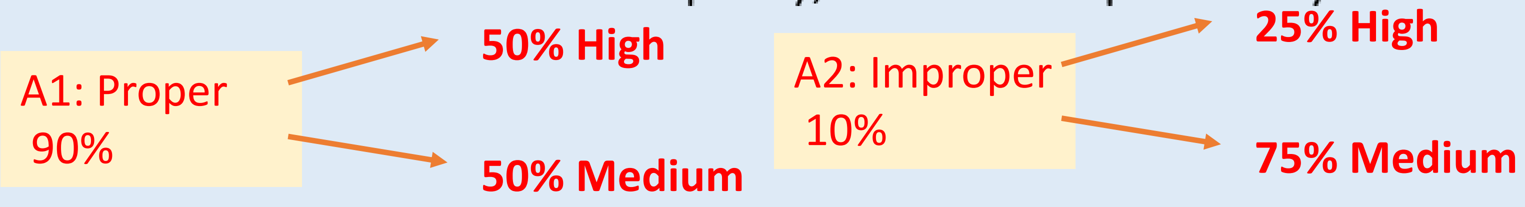
$$\mathbf{P(D) = 0.0034}$$

$$P(A_1/D) = \frac{P(A_1)\,P(D/A_1)}{P(D)} = \frac{(0.1)\,(0.001)}{(0.0034)} = \frac{0.0001}{0.0034} = \frac{1}{34}$$

EXAMPLE: Suppose that when a machine is adjusted properly, 50% of the items produced by it are of high quality and the other 50% are of medium quality. Suppose, however, that the machine is improperly adjusted during 10% of the time and that under these conditions 25% of the items produced by it are of high quality and 75% are of medium quality.

a. Suppose that one item produced by the machine is selected at random, find the probability that it is of medium quality.

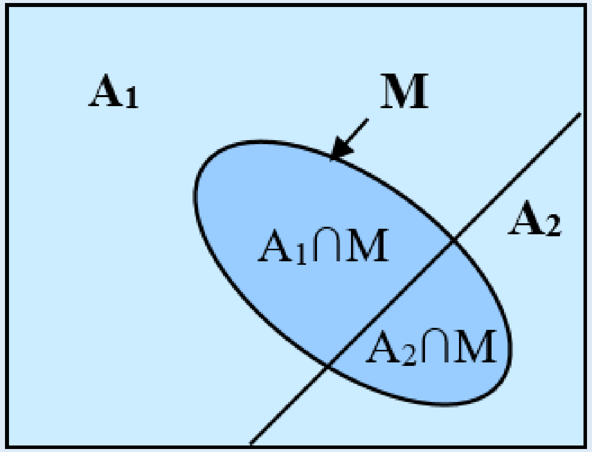
b. If one item is selected at random and found to be of medium quality, what is the probability that the machine was adjusted properly?



SOLUTION:

A_1 = Event machine is properly adjusted, A_2 = Event machine is improperly adjusted

H = Event item is of high quality, M = Event item is of medium quality



From the problem statement we have:

$P(A_1) = 0.9$; $P(A_2) = 0.1$, $P(H/A_1) = 0.5$; $P(H/A_2) = 0.25$

$P(M/A_1) = 0.5$; $P(M/A_2) = 0.75$

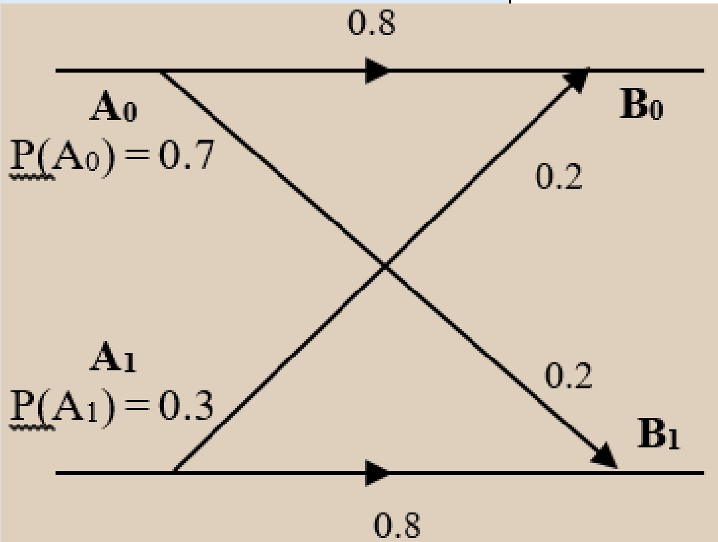
$$P(M) = P(A_1 \cap M) + P(A_2 \cap M) \quad P(M) = P(A_1)\,P(M/A_1) + P(A_2)\,P(M/A_2)$$

$$\boldsymbol{P(M) = (0.9)(0.5) + (0.1)(0.75) = 0.525}$$

$$P(A_1/M) = \frac{P(A_1 \cap M)}{P(M)} = \frac{P(A_1)\,P(M/A_1)}{P(M)} \Rightarrow P(A_1/M) = \frac{(0.9)(0.5)}{(0.525)} = 0.8571$$

EXAMPLE: Consider the problem of transmitting binary data over a noisy communication channel. Due to the presence of noise, a certain amount of transmission error is introduced. Suppose that the probability of transmitting a binary 0 is 0.7 (70% of transmitted digits are zeros) and there is a 0.8 probability that a given 0 or 1 being received properly.

a. What is the probability of receiving a binary 1?
b. If a 1 is received, what is the probability that a 0 was sent?



SOLUTION:

A_0 = Event 0 is sent, A_1 = Event 1 is sent
B_0 = Event 0 is received, B_1 = Event 1 is received

From the problem statement we have:

$P(A_0) = 0.7$; $P(A_1) = 0.3$, $P(B_0/A_0) = 0.8$; $P(B_0/A_1) = 0.2$
$P(B_1/A_0) = 0.2$; $P(B_1/A_1) = 0.8$

a- $P(B_1) = P(A_0)\,P(B_1/A_0) + P(A_1)\,P(B_1/A_1)$
$\boldsymbol{P(B_1) = (0.7)(0.2) + (0.3)(0.8) = 0.38}$
$\boldsymbol{P(B_0) = 1 - P(B_1) = 0.62}$

b- $$P(A_0/B_1) = \frac{P(A_0 \cap B_1)}{P(B_1)} = \frac{P(A_0)\,P(B_1/A_0)}{P(B_1)} = P(A_0/B_1) = \frac{(0.7)(0.2)}{(0.38)} = 0.3684$$

EXAMPLE: A coin may be fair or it may have two heads. We toss it (n) times and it comes up heads on each occasion. If our initial judgment was that both options for the coin (fair or both sides heads) were equally likely (probable), what is our revised judgment in the light of the data?

SOLUTION:

Let A : event representing coin is fair

B : event representing coin with two heads

C : outcome of the experiment $\underbrace{H\,H\,H\,H\,H \ldots H}_{\text{n times}}$

n	P(A/C)
1	1/3
2	1/5
3	1/9
4	1/17
5	1/33

A priori probabilities:

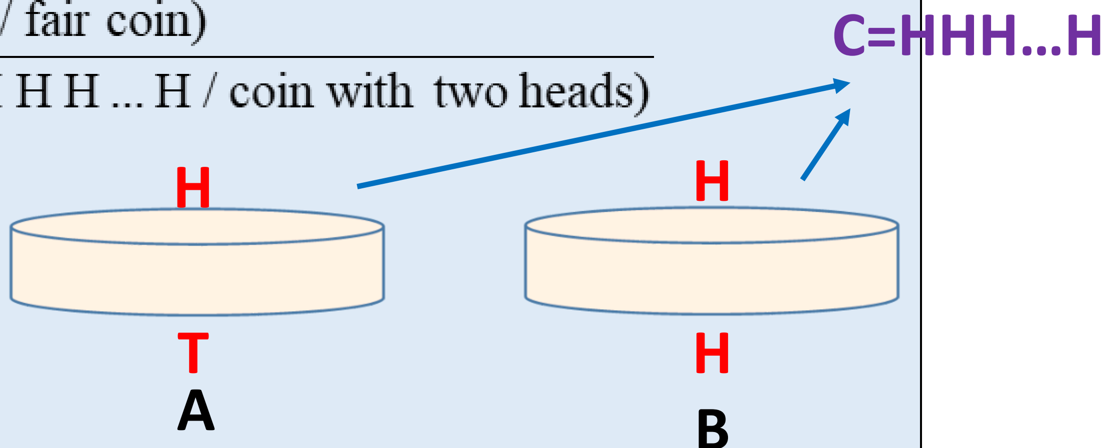
$P(A) = 1/2 \quad , \quad P(B) = 1/2$

➔ We need to find $P(A/C) = ?$

$$P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A)\,P(C/A)}{P(C)}$$

$$P(A/C) = \frac{P(A)\,P(H\,H\,H \ldots H / \text{fair coin})}{P(A)\,P(H\,H\,H \ldots H / \text{fair coin}) + P(B)\,P(H\,H\,H \ldots H / \text{coin with two heads})}$$

$$P(A/C) = \frac{\frac{1}{2}\left(\frac{1}{2}\right)^n}{\frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{1}{2}(1)} = \frac{\left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}\right)^n + 1} = \frac{1}{1+2^n}$$



P(A)=1/2. Find P(A/C)

$$P(B/C) = 1 - P(A/C) = 1 - \frac{1}{1+2^n} = \frac{2^n}{1+2^n}$$

Counting Techniques for Simple Sample Spaces

Here we introduce systematic counting of sample points in a sample space. This is necessary for computing the probability P(A) in experiments with a finite sample space (S) consisting of (n) equally likely outcomes. Here, each outcome has a probability $\left(\frac{1}{n}\right)$.

$$P(A) = \frac{\textit{Number of elements in A}}{\textit{Number of elements in S}}$$

If A is an event defined on S and consists of (m) outcomes, then $P(A) = \frac{m}{n}$

Multiplication Rule:

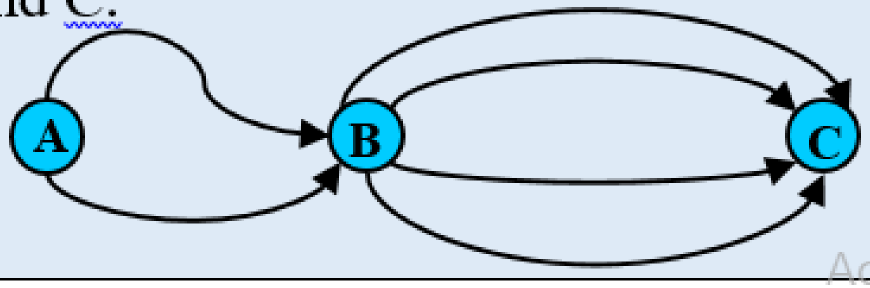
If operation A can be performed in n_1 different ways and operation B in n_2 different ways, then the sequence (operation A, operation B) can be performed in $n_1 \times n_2$ different ways.

EXAMPLE

There are two roads between towns A and B and four roads between towns B and C. How many different routes can one travel between A and C.

SOLUTION

$n = 2 \times 4 = 8$

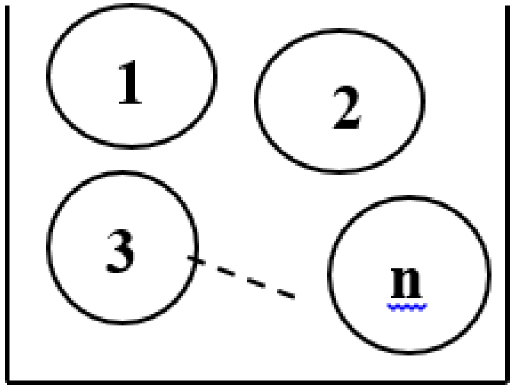


Counting Techniques: A General Setup

Consider an urn containing (n) distinguishable objects (numbered 1 to n). We perform the following experiments

1- Sampling without replacement (order important, repetition not allowed)

An object is drawn; its number is recorded and then put aside. Another object is drawn; its number is recorded and then put aside. The process is repeated (k) times. The total number of ordered sequences $\{x_1, x_2, \ldots, x_k\}$ *(repetition is not allowed)* called **permutation** is:



$$N = n\,(n-1)\,(n-2)\,\ldots\,(n-k+1)$$

$$N = n(n-1)(n-2)\ldots(n-k-1)\frac{(n-k)(n-k-1)\ldots(3)(2)(1)}{(n-k)(n-k-1)\ldots(3)(2)(1)} = \frac{n!}{(n-k)!}$$

Where, $n! = n\,(n-1)\,(n-2)\,\ldots\,(3)\,(2)\,(1)$

2- Sampling with replacement (order important, repetition allowed)

An object is drawn; its number is recorded and then dropped back into the urn. Another object is drawn; its number is recorded and then dropped back. The process is repeated (k) times. The number of possible sequences $\{x_1, x_2, \ldots, x_k\}$ of length (k) that can be formed from the set of (n) distinct objects *(repetition allowed)*: $N = n^k$

n	n-1	n-2	n-3	n-k+1
n	n	n	n	n

3- Sampling without replacement (order not important, repetition not allowed)

Here, k objects are selected as one group from the n objects. The order within the selected group is not important. The number of **combinations** of (n) different objects, taken (k) at a time, without repetition is the number of sets that can be made up from the (n) given objects, each set containing (k) different objects and no two sets containing exactly the same (k) objects.

The number is:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

3	2	1	2	3	1
3	1	2	1	2	3
2	1	3	1	3	2

When the order is not important all 3! Permutations become one combination (k!)

Note that:

$$\underbrace{\left(\begin{array}{c}\text{Choose (k) objects from}\\ \text{(n), order not important}\end{array}\right)}_{N} \text{ same as } \underbrace{\left(\begin{array}{c}\text{First select (k) objects}\\ \text{from (n), order important}\end{array}\right)}_{\frac{n!}{(n-k)!}} \text{ then } \underbrace{\left(\begin{array}{c}\text{Divide by the different}\\ \text{arrangements of k objects}\end{array}\right)}_{k!}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{\frac{n!}{(n-k)!}}{k!}$$

EXAMPLE: How many different five-letter computer passwords can be formed

a- If a letter can be used more than once.

b- If each word contains a letter no more than once.

SOLUTION:

a- $N = (26)^5$; **Sampling with replacement, order important.**

b- $N = \dfrac{26!}{(26-5)!}$; **Sampling without replacement, order important**

a	a	a	a	a
a	a	a	a	b
a	a	a	b	a
a	b	a	a	a
.	.	.	.	.
z	z	z	z	z

26 letters

a, b, c, ,x, y, z

In practice, a password contains Uppercase, lowercase, strings and numbers. Such a password is difficult to break.

26	26	26	26	26	With
26	25	24	23	22	Without

Counting Techniques

EXAMPLE: From four persons (set of elements), how many committees (subsets) of two members (elements) may be chosen?

SOLUTION:

Let the persons be identified by the initials A, B, C and D. Let S = {A, B, C, D}

Subsets: (A , B) , (A , C) , (A , D) , (B , C) , (B , D) , (C , D); **6 subsets**

$$N = \binom{4}{2} = \frac{4!}{2!\ (4-2)!} = 6\ ; \quad \textbf{6 subsets}$$

Missing sequences: (A , A) , (B , B) , (C , C) , (D , D) ➔ (repetition is not allowed)

Missing sequences: (B , A) , (C , A) , (D , A)
(C , B) , (D , B) , (D , C) ➔ (order is not important)

EXAMPLE: A box contains six cards marked I to 6. We draw two items subject to the following sampling techniques:

SOLUTION: n = 6 and k = 2

a- Sampling with replacement, repetition allowed, order important

$N = n^k = 6^2 = 36$ (all elements red + black +blue)

b- Sampling without replacement repetition not allowed, order important (Permutations)

$$N = \frac{n!}{(n-k)!} = \frac{6!}{(6-2)!} = 30$$ (red + blue.

Black diagonal elements are removed since they represent repetition elements)

D2 / D1	1	2	3	4	5	6
1	*(1,1)*	**(1,2)**	**(1,3)**	**(1,4)**	**(1,5)**	**(1,6)**
2	**(2,1)**	*(2,2)*	**(2,3)**	**(2,4)**	**(2,5)**	**(2,6)**
3	**(3,1)**	**(3,2)**	*(3,3)*	**(3,4)**	**(3,5)**	**(3,6)**
4	**(4,1)**	**(4,2)**	**(4,3)**	*(4,4)*	**(4,5)**	**(4,6)**
5	**(5,1)**	**(5,2)**	**(5,3)**	**(5,4)**	*(5,5)*	**(5,6)**
6	**(6,1)**	**(6,2)**	**(6,3)**	**(6,4)**	**(6,5)**	*(6,6)*

c. Sampling without replacement, repetition not allowed, order not important (Combinations)

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{6!}{2!(6-2)!} = 15$$ (either red or blue but not both since the order is not important.

Diagonal back elements are removed as well since repetition is not allowed).

Arrangement of Elements of Two Distinct Types: The Binomial Coefficient

When a set contains only elements of two distinct types, type (1) consists of k elements and type (2) consists of (n-k) elements, then the number of different arrangements of all the elements in the set is given by the binomial coefficient $\binom{n}{k}$

Example: How many different binary numbers of five digits can be formed from the digits 1, 0 such that each number contains two ones? List these numbers.

Solution: Here, n = 5, k = 2 and the number of different combinations is:

$$\binom{5}{2} = \frac{5!}{3!2!} = \frac{(5)(4)(3)(2)(1)}{(3)(2)(1)(2)(1)} = 10$$

1	1	0	0	0
0	1	1	0	0
1	0	0	1	0
.	.	.	.	.
1	0	0	0	1

Let us pose question as: In how many ways can we arrange 5 different balls in a row?
Answer: (5)!

Arrangement of Elements of k Distinct Types: :The Multinomial Coefficient

The number of ways to arrange n items of which n_1 are of one type, n_2 of a second type, …, n_k of a k'th type is given by $N = \begin{pmatrix} & & n & & \\ n_1 & n_2 & . & . & n_k \end{pmatrix} = \dfrac{n!}{n_1!n_2!\ldots n_k!}$

Example: In how many ways can we arrange 5 red balls, 3 green balls, and 2 white balls in a row?

Solution: $N = \begin{pmatrix} & & n & & \\ n_1 & n_2 & . & . & n_k \end{pmatrix} = \begin{pmatrix} & 10 & \\ 5 & 3 & 2 \end{pmatrix} = \dfrac{10!}{5!3!2!}$

Let us pose question as: In how many ways can we arrange 10 different balls in a row?

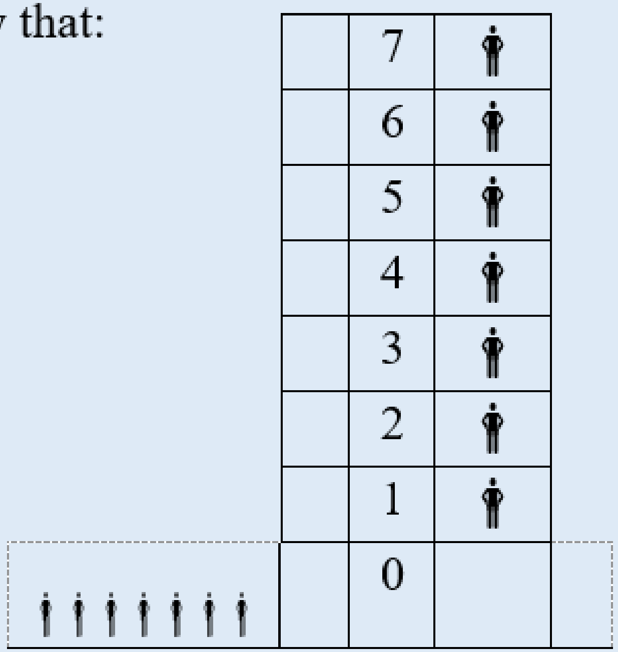
Answer: (10)!

R	R	R	R	R	G	G	G	W	W

EXAMPLE: An apartment building has eight floors (numbered 0 to 7). If seven people get on the elevator on the ground floor, what is the probability that:

a- all passengers get off on different floors?

b- all passengers get off on the same floor?



SOLUTION

Number of points in the sample space:
First person can get off at any of the 7 floors.
Second person can get off at any of the 7 floors and so on.

➔ The number of ways people can get off:

(N) $= 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 = 7^7$

a- Here the problem is to find the number of permutations of 7 objects taking 7 at a time.

$$P = \frac{\text{number of ways passengers get off on different floors}}{\text{total number of ways passengers can get off}} = \frac{(7)(6)(5)(4)(3)(2)(1)}{7^7} = \frac{7\,!}{7^7}$$

b- Here there are 7 ways whereby all seven persons get off on the same floor.

$$P = \frac{\text{number of ways all passengers get off on the same floor}}{\text{total number of ways passengers can get off}} = \frac{7}{7^7}$$

Note that the classical definition of probability was used in parts a and b assuming that all points in the sample space are equally likely. $P(A) = \dfrac{\text{Number of elements in A}}{\text{Number of elements in S}}$

EXAMPLE: To determine an "**odd man out**", each one of n players tosses a fair coin. If one player's coin turns up differently from all the others, that person is declared the odd man out and the game ends. Let (A) be the event that one person is declared an odd man out.

a- Find P(A)

b- Find the probability that the game is terminated with an odd man out after (k) trials

SOLUTION:

a- $$P(A) = \frac{\text{Number of outcomes in event (A)}}{\text{Number of possible sequences}}$$

number of outcomes leading to an odd man out:

(n – 1) Heads and one Tail

(n – 1) Tails and one Head

$$P(A) = 2\frac{n}{2^n} = \frac{n}{2^{n-1}}$$

with an odd man out, a success is obtained and the game is over.

b- A second trial is needed when the experiment ends with a failure:

➔ P(a second trial is needed) = 1 – P(A)

For (k) trials:

$$P(\underbrace{F\,F\,F\ldots F}_{k-1\ Trials}\ \underline{\underline{S}}) = P(F)^{k-1}\,P(S)$$

$$P(\underbrace{F\,F\,F\ldots F}_{k-1\ Trials}\ \underline{\underline{S}}) = [1 - P(A)]^{k-1}\,P(A)$$

$\overbrace{\text{O O O O O} \ldots \text{O}}^{n}$ **n≥3**

H/T H/T H/T H/T H/T … H/T

$$\left.\begin{matrix} H\,H\,H\,H\,H \ldots T \\ H\,H\,H\,H \ldots T\,H \\ H\,H\,H \ldots T\,H\,H \\ \vdots \\ T\,H\,H\,H\,H \ldots H \end{matrix}\right\} \Rightarrow (n)$$

$$\left.\begin{matrix} T\,T\,T\,T\,T \ldots H \\ T\,T\,T\,T \ldots H\,T \\ T\,T\,T \ldots H\,T\,T \\ \vdots \\ H\,T\,T\,T\,T \ldots T \end{matrix}\right\} \Rightarrow (n)$$

n	P(A)
3	6/8
4	4/8
5	5/16
6	6/32